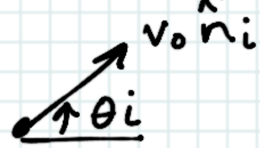
C i) Interactions I: alignment + flocking

Birds flocking. Craig Reynolds 1987 "boids"

e.g. Batman Returns

Vicsek 1995; a lot like the x-y model for ferromagnets; except objects break time-reversal symmetry because they move.

Each agent has a preferred "polarization" direction:



$$\hat{n}_i = (\cos\theta_i, \sin\theta_i)$$

- overdamped dynamics
- N point particles
- move at fixed speed v_0

$$\frac{d\vec{r}_i}{dt} = v_0 \hat{n}_i \quad \Rightarrow \quad \begin{aligned} \vec{r}_i(t+\Delta t) &= r_i(t) + v_0 \hat{n}\, \Delta t \\ \theta_i(t+\Delta t) &= \sum_j \theta_j(t) + \eta_i(t) \end{aligned}$$

$\uparrow$ neighbors within a distance R

η_i random uniform

Simulations: for large enough v_0 + small enough noise spontaneous formation of flocks.

$\boxed{\text{first}}$ order or second order transition?

$\uparrow$ very difficult; finite-size scaling of lots of particles

Our goal: understand why the system becomes unstable

↳ most fluctuations are stable (decay quickly)

↳ some decay slowly or even grow

↳ develop a hydrodynamic theory that focuses on the slow degrees of freedom

technically, relaxation rates $\omega(q)$ relax as $q \to 0$.

∃ different ways of developing such theories:

① derive by coarse-graining microscopic dynamics

CONS: typically need to be in a low-density regime OR make assumptions about higher-order correlations

?★ PROS: Can connect hydrodynamic model parameters to the microscopic models

② phenomenological: identify fields and write down couplings allowed by symmetry

CONS: no connection to microscopics

PROS: possible to do even for difficult problems

③ near equilibrium entropy productions

CONS: not clear how it works far from equilibrium

PROS: can handle complicated interactions

Reference:

Hydrodynamics of soft active matter.
Marchetti et al Rev. Mod. Phys
85 1143 (2013)

(A) Which fields have $\omega(q) \to 0$ as $q \to 0$?
relaxation rates

1) local densities of conserved quantities

2) "broken-symmetry" Goldstone modes that have no restoring force at zero wavenumber

3) near a continuous phase transition, the amplitude of the order parameter.

this is why it is useful to classify broken symmetry
"polar"
"apolar"

and whether momentum is conserved
↳ no in systems with drag
↳ yes in systems with hydrodynamic

For Vicsek model, suggests two slow fields

$$\rho(\vec{r}, t) = \sum_n \delta(\vec{r} - \vec{r}_n(t))$$

$$\vec{p}(\vec{r}, t) = \frac{1}{\rho(\vec{r}, t)} \sum_n \hat{v}_n \, \delta(\vec{r} - r_n(t))$$

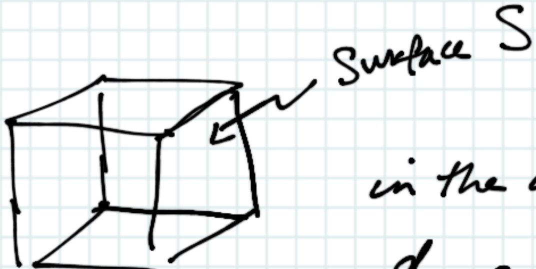
instantaneous orientation of each particle

Toner-Tu continuum eqn for Vicsek model (PRL 75) 1995

Assert: conserved density of "boids" $\rho \Rightarrow \frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\left(v_0 \rho \vec{P}\right) = 0$

$\vec{P}$ is the field that describes <u>both</u> the locally averaged velocity orientation of the flock and the local order parameter of alignment.

→ so this is just a continuity equation: the density changes at a rate $v_0\vec{P}$:



so flux is $j = \rho v_0 \vec{P}$

in the absence of sources or sinks

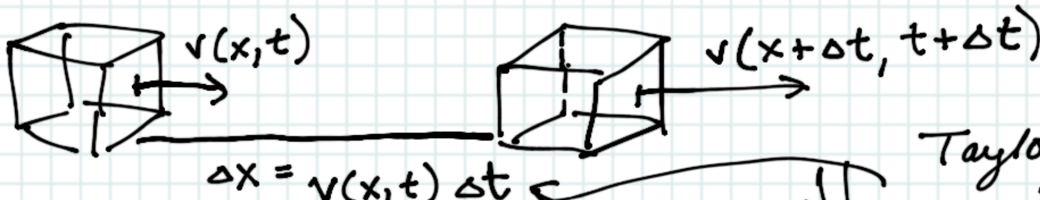
$$\frac{d}{dt}\int \rho \, dV = -\int_S j \cdot dS$$

$\Downarrow$ divergence theorem

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(v_0 \rho \vec{P}\right) = 0$$

Now for $\vec{P}$:

First, advective derivative



$\Downarrow$ Taylor expansion + substitution

$$\Delta v(x,t) = \frac{\partial v}{\partial x}\left(v(x,t)\Delta t\right) + \frac{\partial v}{\partial t}\Delta t = \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) v$$

in higher dimensions: $\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v}$

True for *any* advected vector field: e.g. $\left(\frac{\partial}{\partial t} + \vec{v}_1 \cdot \vec{\nabla}\right)\vec{p}$

What is the velocity? due to motion with respect to substrate, galilean invariance is broken, so mean velocity is $v \neq v_0$

under a boost to a reference frame at constant velocity

$\vec{v} = v\vec{p}$ this is an emergent parameter. at low densities $v \sim \frac{v_0}{2}$

$$\underbrace{\partial_t \vec{p} + v_1(\vec{p}\cdot\vec{\nabla})\vec{p}}_{\text{advective derivative}} = -\frac{1}{\gamma}\frac{\partial F_p}{\partial \vec{p}} + f$$

at high densities can become very low.

white noise

$$\langle f_\alpha(\vec{r},t), f_\beta(\vec{r}',t')\rangle = 2\Delta\,\delta_{\alpha\beta}\,\delta(\vec{r}'-r)\,\delta(t-t')$$

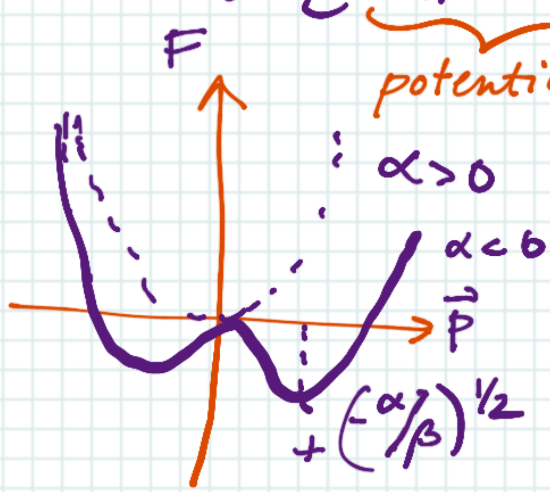
This term can be thought of as coming from the free energy of a ferro electric liquid crystal:

Frank energy (liquid crystals)

$$\mathcal{F} = \int\Big[\underbrace{\frac{\alpha}{2}|\vec{p}|^2 + \frac{\beta}{4}|\vec{p}|^4}_{\text{potential that allows broken symmetry}} + \frac{K}{2}\underbrace{(\partial_\alpha p_\beta)^2}_{\text{penalty for splay + bend perturbations}} - v_1\,\vec{\nabla}\cdot\vec{p}\,\frac{\delta\rho}{\rho_0} \underbrace{+ \frac{\lambda}{2}|\vec{p}|^2\,\vec{\nabla}\cdot\vec{p}}_{\text{coupling of } \rho + p \text{ to splay}}$$

$\rho - \rho_0$

$p_\| p_\perp$



Then the LHS becomes

$$-\left[\alpha(\rho) + \beta|p|^2\right]\vec{p} + K\nabla^2\vec{p} - v_1 \frac{\nabla\rho}{\rho_0}$$
$$+ \frac{\lambda}{2}\nabla|p|^2 - \lambda\vec{p}(\vec{\nabla}\cdot\vec{p}) + f$$

Note: there are two additional terms allowed by symmetry

$$\frac{\lambda_3}{2}\nabla|p|^2 \text{ and } \lambda_2\vec{p}\,(\vec{\nabla}\cdot\vec{p})$$

Where are the "relevant" terms allowed by "symmetry"?

"relevant": large length + long timescales $\Rightarrow$ lowest order gradients in time + space

: gradient expansion

"symmetries": rotational invariance

space + time translation invariance

~~Galilean invariance~~ (speeds of all birds will not remain constant after a boost)

if it did hold $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$

$$\partial_t\vec{v} + \lambda_1(\vec{v}\cdot\nabla)\vec{v} + \lambda_2(\nabla\cdot\vec{v})\vec{v} + \lambda_3\vec{\nabla}(|\vec{v}|^2)$$

$$= \underbrace{D_1\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_2\nabla^2\vec{v} + D_3(\vec{v}\cdot\vec{\nabla})^2\vec{v}}_{2^{nd}\text{ order spatial derivatives}} + \alpha\vec{v} - \beta|\vec{v}^2|\vec{v} + \nabla\rho$$

vector $\vec{v}$; scalars $|\vec{v}|^2$ and ρ

alternatively, one can consider in analogy to Navier-Stokes

$K\nabla^2\vec{p}$ ~ viscous damping

$-v_1\nabla\frac{f}{\rho_0} + \frac{\lambda}{2}\vec{\nabla}|\vec{p}|^2 - \lambda p(\nabla\cdot p)$ ~ $-(1/\rho_0)\nabla P$, $P \simeq v_1\rho - \frac{\lambda\rho_0}{2}|p|^2$

$-[\alpha + \beta|\vec{p}|^2]\vec{p}$ ~ nonlinear friction

Two questions:

(a) Can this be derived from microscopic Vicsek model?

(b) What do these equations tell us about system?

(a) Yes. See Bertin et. al (2006) PRE 74 or Ihle (2011) PRE 83

General idea: define a joint microscopic density

$$\Psi(\vec{r},\theta,t) = \sum_{i=1}^{N} \delta(\vec{r}-\vec{r}_i)\,\delta[\theta-\theta_i(t)]$$

because the equations for $\dot{\vec{r}}$, $\dot{\theta}$ are Langevin (have white noise)

$\exists$ a formal solution for $\partial_t\Psi$

$$\partial_t\Psi = \partial_\theta\Big[-\mu_r\,\underbrace{\Psi(r,\theta)\int_\theta\int \Psi(x,\phi)\,\overbrace{T(\theta-\phi,r-x)}^{\text{torque interactions b/w spins}}\,d\vec{x}\,d\phi}_{\text{couples } \vec{r} \text{ to } \vec{x} \text{ through interactions}} + D_r\partial_\theta\Psi + (2D_r\Psi)^{1/2}\,\Gamma\Big]$$

Γ: Gaussian white noise

$$\rho(\vec{r},t) = \int \langle \Psi(\vec{r},\theta,t) \rangle \, d\theta$$ average density

$$p(\vec{r},t) = \int \hat{e}(\theta) \langle \Psi(r,\theta,t) \rangle \, d\theta$$

Problem: to compute $\langle \Psi \rangle$, have a hierarchy of equations that couples, for example, one-particle density function to two-particle density functions.

→ to close, need to make an assumption about joint probability distributions

a la "molecular chaos" to derive Boltzmann

e.g. $\Psi_2(\vec{r},\vec{x},\theta,\phi) \simeq \Psi_1(\vec{r},\theta)\Psi_1(x,\phi)$

↳ end result

$$\alpha = D_r - \frac{1}{2\pi}\gamma\rho \Rightarrow \rho_c = \frac{2D_r\pi}{\gamma}$$ $\alpha_c = 0$ } Doesn't quite work

$$\beta = \frac{\gamma^2\rho^2}{32 D_r \pi^2}$$

so

$$\alpha = D_r\left(1 - \frac{\rho}{\rho_c}\right)$$

ρ_c fit param.

See explicit derivation in review by by Marchetti et al Section B.3 (p 42)

(b) What does this hydrodynamic theory tell us about the system?

- mean field: by construction, ϕ^4-like theory
$$\begin{cases} \text{density } \rho_0 < \rho_c : \text{system is isotropic} \\ \text{density } \rho_0 > \rho_c : \text{system is ordered with } |\vec{p}_0| = \sqrt{\frac{a(\rho_0)}{b}} \end{cases}$$

- Mermin-Wagner theorem forbids spontaneous symmetry breaking in $d=2$ equilibrium system (Mermin Wagner 1966, Chaikin + Lubensky 2000)

- is this because there should be fluctuation corrections (requiring RG analysis)?

No. Toner and Tu demonstrate using RG the non-equilibrium model is different for $d < d_c = 4$, but $\underline{\exists}$ broken continuous symmetry even in $d=2$

Why? the advective derivative generates effective long-ranged interactions. A spin's neighbors will be different at some later time depending on the velocity field, allowing two distant spins to interact.

Two examples of what we can learn about states from hydrodynamic theory:

① fluctuations about isotropic case

$$\delta\rho = \rho - \rho_0$$
$$\delta\vec{p} = \vec{p} - p_0$$

linearized EOM:

$$\partial_t \delta\rho = -v_0\rho_0\vec{\nabla}\cdot\vec{p}$$
$$\partial_t \vec{p} = -\alpha_0\vec{p} - \frac{v_1}{\rho_0}\nabla\delta\rho + K\nabla^2\vec{p} + \vec{f}$$

Assume $\rho = \sum_q \underbrace{A_q e^{i(\vec{q}\cdot\vec{r} - \omega t)}}_{\rho(q,\omega)}$ $\quad \vec{p} = \sum_q B_q e^{i(\vec{q}\cdot\vec{r} - \omega t)}$

$$-i\omega\tilde{\rho}(q\omega) = -v_0\rho_0 i\vec{q}\cdot\tilde{\vec{p}} \Rightarrow \tilde{\rho} = \frac{v_0\rho_0\vec{q}\cdot\tilde{\vec{p}}}{\omega}$$

$$-i\omega\tilde{\vec{p}}(q,\omega) = -\alpha_0\tilde{\vec{p}} \qquad -\frac{i\vec{q}\tilde{\rho}v_1}{\rho_0} - Kq^2\tilde{\vec{p}}$$

skip

longitudinal

$$-i\omega = -\alpha_0 - \frac{i\vec{q}v_1}{\rho_0}\left(\frac{v_0\rho_0}{\omega}\vec{q}\right) + Kq^2$$

$$-i\omega^2 = (-\alpha_0 - Kq^2)\omega - i q^2 v_1 v_0$$

$$0 = \omega^2 + i\omega(\alpha_0 + Kq^2) + q^2 v_1 v_0$$

$\therefore$

$$\omega(q) = -\frac{i}{2}(\alpha_0 + Kq^2) \pm \frac{i}{2}\sqrt{(\alpha_0 + Kq^2)^2 - 4q^2 v_0 v_1}$$

Linear stability: $\mathrm{Im}[\omega(q)] < 0 \Rightarrow v_0 v_1 > 0$

★ positive, $< (\alpha_0 + Kq^2)^2$. if $v_1 \to 0$, could be unstable (MIPS)

When is $\mathrm{Re}\,[\omega(q)] \neq 0$? $\alpha_0 \leq \dfrac{v_0 v_1}{K}$

★ becomes negative.

near $\alpha_0 \to 0^+$

can show $q \sim 2\sqrt{v_0 v_1}/K$.

and $\omega(q) = \pm q\sqrt{v_0 v_1}$ $\Rightarrow$ like a propogating sound mode in a crystal with $\omega \sim cq$

propogating density wave near $\alpha_0 \to 0^+$

② fluctuations about ordered state

idea: $\vec{p} = p\hat{n}$

$$\delta\vec{p} = \hat{n}_0\,\delta p + p_0\,\delta\vec{n}$$

WLOG $\hat{n}_0 = \hat{x}$; $|\hat{n}|^2 = 1 \Rightarrow \hat{n}_0 \cdot \delta\vec{n} = 0 \Rightarrow \delta\vec{n} = \delta n\,\hat{y}$

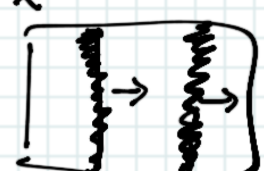
$$\delta\vec{p} = \hat{x}\,\delta p + \hat{y}\,p_0\,\delta n$$

write linearized equations + fourier transform

$$-i\omega \begin{bmatrix} \tilde{\delta\rho} \\ \tilde{\delta p} \\ \tilde{\delta n} \end{bmatrix} = \underbrace{\vec{M}(\vec{q})}_{3\times 3\ \text{matrix}} \cdot \begin{bmatrix} \tilde{\delta\rho} \\ \tilde{\delta p} \\ \tilde{\delta n} \end{bmatrix} + F(q)$$

; look for when $\mathrm{Im}(\omega) < 0$

i) for $\alpha_0 \to 0^-$, find ordered bands aligned transverse to broken symmetry + propogating in $\hat{x}$



ii) $\alpha_0 \ll -1$ δp decays fast
and coupled eqns for δn, δp give
"sound" modes with $\omega_\pm = q\, c_\pm(\theta) - i q^2 \kappa_\pm(\theta)$
angle between $\vec{q}$ and $\hat{x}$.

signature of δn becoming "massless": no restoring force to director fluctuations
$\Rightarrow$ large density fluctuations are allowed
"giant number fluctuations"