1 Phys 576, HW Problem Set 6

1. (a) Consider a dielectric crystal made of layers of atoms, with rigid coupling between layers so that the motion of the atoms is restricted to the plane of the layer. Show that the phonon heat capacity in the Debye approximation is the low temperature limit is proportional to $T^2$.

(b) Suppose instead, as in may layer structures, that the adjacent layers are very weakly bound to each other. What form would you expect the phonon heat capacity to approach at extremely low temperatures?

2. Elasticity explored

The stress tensor $\sigma_{\alpha\beta}$ can be defined by the following expression:

$$U = \frac{1}{2} \sum_{\alpha\beta} \int d\vec{r} \epsilon_{\alpha\beta} \sigma_{\alpha\beta},$$

where $\epsilon_{\alpha\beta}$ is the symmeterized strain tensor discussed in class. This is the generalization of the “work equals force times distance” rule.

(a) Write an expression for the stress tensor in terms of the elastic tensor $C_{\alpha\beta\gamma\delta}$.

As discussed in class, the energy for a solid with cubic symmetry can be simplified due to the symmetries (because there are only three elastic constants):

$$U = \frac{1}{2} \int d\vec{r} \{ C_{xxxx}[\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2] + 2C_{xxyy}[\epsilon_{xx}\epsilon_{yy} + \epsilon_{xx}\epsilon_{zz} + \epsilon_{zz}\epsilon_{yy}] + 4C_{xyxy}[\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{zx}^2] \}.$$

Many solids (such as glasses) are isotropic, which has even more symmetries than the cubic crystal. For example, in glasses the energy $U$ should remain invariant under 45 degree rotations of the material about the z-axis:

$$\epsilon_{\alpha\beta}(\vec{r}) = \sum_{\gamma\delta} R_{\alpha\gamma}^{*} \epsilon_{\gamma\delta}(\vec{r}^{\prime}) R_{\delta\beta},$$

where $R$ is a matrix that represents a rotation about the z-axis by 45 degrees.

(b) write an expression for the 3 by 3 matrix $R$.

(c) Substitute this last equation into the previous two, and demand that the energy vanish. Show that this requires that:

$$0 = (2C_{xyxy} + C_{xxyy} - C_{xxxx})(\epsilon_{yy} - 2\epsilon_{xy} - \epsilon_{xx})(\epsilon_{yy} + 2\epsilon_{xy} - \epsilon_{xx}).$$

and argue that this must be true:

$$2C_{xyxy} + C_{xxyy} = C_{xxxx}.$$

d) Substitute this back in to the expression for the energy to show that the energy of an isotropic solid can be written as:

$$U = \frac{1}{2} \int d\vec{r} \lambda (\sum_{\alpha} \epsilon_{\alpha\alpha})^2 + 2\mu \sum_{\alpha\beta} \epsilon_{\alpha\beta}^2.$$

What are $\lambda$ and $\mu$ in terms of the elastic constants $C_{\alpha\beta\gamma\delta}$? These constants are called the “Lame constants”.

e) Extra credit: (+4 points out of 20) Show that the variational derivative of the total energy $U$ with respect to the displacements can be written:

$$\frac{\delta U}{\delta u_{\alpha}(\vec{r})} = \sum_{\beta} \frac{\partial}{\partial r_{\beta}} \sigma_{\alpha\beta}(\vec{r}).$$

This is very useful, as the equation of motion for a variable $\rho \ddot{u}_{\alpha}(\vec{r}) = -\frac{\delta U}{\delta u_{\alpha}(\vec{r})}$. 

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