1. **Low temp specific heat in d-dimensions**
   (a) Show that the density of normal modes in the Debye approximation gives the exact (within the harmonic approximation) leading low-frequency behavior of $g(\omega)$, provided that the velocity $c$ is taken to be:

   \[
   \frac{1}{c^3} = \frac{1}{3} \sum_s \int \frac{d\Omega}{4\pi} \frac{1}{c_s(q)^3}
   \]  

   (b) Show that in a d-dimensional harmonic crystal, the low-frequency density of normal modes varies as $\omega^{d-1}$.
   (c) Deduce from this that the low-temperature specific heat of a harmonic crystal vanishes as $T^d$ in $d$ dimensions.

2. **Anomalous density of states**
   A certain two-dimensional simple-square lattice of lattice constant $a$ has the dispersion relation
   \[
   (\omega_L(k))^2 = c_L^2 k^2, \quad \text{for longitudinal vibrations and}
   (\omega_T(k))^2 = c_T^2 k^2 \quad \text{for transverse vibrations for} \quad ka << 1.
   \]
   The density of mode frequencies $g(\omega)$ determines the temperature dependence of the specific heat.
   a) What is $g(\omega)$ for an N-atom crystal at frequencies $\omega$ described by the dispersion relation above? That is, what is the number of modes between $\omega$ and $\omega + d\omega$?
   Suppose that for some wave-vector $\vec{k}_0$, the $\omega_L(k)$ has an absolute maximum. Near this maximum $\omega_L(k) = \omega_0 - A(\vec{k} - \vec{k}_0)^2$. You can assume $\omega_T(k)$ always lies well below $\omega_0$, so that these modes dont contribute to $b$).
   b) Find the form of $g(\omega)$ when $\omega \leq \omega_0$, and when $\omega \geq \omega_0$.

3. **van Hove singularities**
   a) In a linear harmonic chain with only nearest-neighbor interactions, the normal-mode dispersion relation has the form $\omega(k) = \omega_0 |\sin(ka/2)|$, where the constant $\omega_0$ is the maximum frequency (assumed when $k$ is on the zone boundary). Show that the density of normal modes in this case is given by:

   \[
   g(\omega) = \frac{2}{\pi a \sqrt{\omega_0^2 - \omega^2}}.
   \]  

   The singularity at $\omega_0 = \omega$ is called a van Hove singularity.
(b) In three dimensions the van Hove singularities are infinities not in the normal mode density itself, but in its derivative. Show that the normal modes in the neighborhood of a maximum of $\omega(\vec{k})$, for example, lead to a term in the normal mode density that varies as $(\omega_0 - \omega)^{1/2}$.

4. **Localization at band edges (this one is supposed to be hard)**

In the alternating-spring system discussed in class (Fig 22.9 in AM) with spring constants given by $K$ and $G$, there is a gap in the density of states i.e., a frequency range where there are no normal modes.

a) Write an expression for the frequencies in the gap. How can one change $K$ and $G$ to make the gap larger or smaller?

Still, we can excite the system at any frequency we like, including a frequency in the gap. Consider a semi-infinite chain with $0 < K - G << K$. Now suppose that the end mass is oscillated with a frequency $\omega^*$ in the middle of the gap, at frequency $\sqrt{(K + G)/M}$.

b) Write the eigenvalue equation for the frequencies (which we derived in class). Show that if we assume $\omega = \omega^*$ (i.e., in the gap), and solve this equation for $k = k^*$, $k^*$ must be complex. Write an expression for the real and complex parts of $k^*$. Expand the expression for the complex part of $k^*$ to second order in $\epsilon = (K - G)/G << 1$.

c) Use this solution to determine the behavior of the amplitudes $u_1 = A_1 \exp[i(\omega t)]$ and $u_2 = A_2 \exp[i(\omega t)]$ as a function of distance from the end.

d) How does the penetration depth grows as $G \to K$ and the gap disappears?