Problem Set 4

1. Consider 1-d chain with identical masses $M$. Assume that there are nearest-neighbor (nn) springs with spring constant $k_1$ and next-nearest neighbor (nnn) springs with spring constant $k_2 < k_1$.
   (a) find the dispersion relation for this system.
   (b) calculate the speed of sound and compare to a system without nnn interactions.

   (c) Now generalize to a new energy:
   $$U^{\text{harmonic}} = \sum_n \sum_{m>0} \frac{1}{2} K_m \left[u(na) - u((n+m)a)\right]^2$$

   (i) Show that the dispersion relation is:
   $$\omega = \frac{2}{\sqrt{m\omega_0}} \frac{K_m}{M} \frac{\sin^2\left(\frac{1}{2}mka\right)}{M}$$

   (ii) Show that the long-wavelength limit of the dispersion relation for $K_m = \frac{K_1}{m^p}$ when $p=3$ is:
   $$\omega \sim k \sqrt{\ln k}$$

2. Consider a linear chain in which alternate ions have mass $M_1$ and $M_2$, and only nearest neighbors interact.

   (a) Show that the dispersion relation for normal modes is:
   $$\omega^2 = \frac{K}{M_1 M_2} \left(\sqrt{M_1^2 + M_2^2 + 2 M_1 M_2 \cos ka}\right)$$

   (b) Discuss the form of the dispersion relation and the nature of the normal modes when $M_1 \gg M_2$.
   (c) Compare the dispersion relation with that of the monatomic linear chain when $M_1 \sim M_2$. 

3. Vibrations of a square lattice (in 2D)

a) Write the equation of motion for the displacement of a lattice point \( u_m \) on a square lattice with ions of equal mass.

b) Assuming a wave of the form:

\[ u = A e^{i (q_x a + q_y ma - \omega t)} \]

calculate the dispersion relation for this lattice.

c) Now assume that the masses are not all equal. Instead, masses on alternating lattice points alternate between M1 and M2. Find the new dispersion relation.

d) Sketch the dispersion curves along a line from the center to the corner of the Brillouin Zone.

4. As discussed in class, the equation of motion for a monatomic 1-D chain is:

\[ M \frac{d^2 u_s}{dt^2} = -K \left[ 2u_s - u_{s+1} - u_{s-1} \right] \]

Show that for long wavelengths this equation reduced to the continuum wave equation, where \( c \) is the velocity of sound:

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]